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"Sample Variance" in Small-Scale CMB Anisotropy Experiments

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ABSTRACT

We discuss the effects of finite sky coverage and the uncertainty in extracting information about the power spectrum from experiments on small angular scales. In general the cosmic variance is enhanced by a factor of $4\pi/A$, where A is the solid angle sampled by the experiment. As a rough guide, an experiment with sensitivity peaking at the ℓth multipole has to cover $\gtrsim \ell$ independent patches to have a lower "sample variance" than for a whole-sky measurement of the quadrupole. Our approach gives a relatively simple way of attaching an error bar to the theoretical prediction for a particular experiment, and thereby comparing theories with experimental results, without the need for computationally-intensive Maximum Likelihood or Monte Carlo calculations.

Subject headings: cosmology: cosmic background radiation — methods: statistical

Introduction

Since the discovery of large angular scale anisotropies in the microwave background (Smoot et al. 1992) effort has concentrated on constraining the primordial power spectrum of fluctuations using the results from degree-scale experiments (e.g. Górski 1992, Vittorio & Silk 1992, Gouda & Sugiyama 1992, Dodelson & Jubas 1993, Górski et al. 1993, White et al. 1993, Crittenden et al. 1993). It has become apparent (e.g. Gould 1993, White et al. 1993) that the "cosmic variance", due to sampling only one universe, is a fundamental limitation on the utility of measurements on the largest scales. This uncertainty comes about because the fluctuations we observe are a single realization of a random variable. The theory can predict properties of the ensemble, but not the individual realizations. For example the quadrupole is drawn from a χ^2 distribution with only 5 degrees of freedom, so that the purely theoretical error on the measured r.m.s. quadrupole is large.

Usually it is assumed that the "cosmic variance" is negligible for small scale ($\lesssim 1^\circ$) experiments which are sensitive only to high multipoles in the $\Delta T/T$ expansion. However, this assumption is only true for a whole-sky measurement. It is clear that the variance from only sampling a fraction of the sky will be larger. We feel that the importance of this "sample variance" has not been emphasized until now.

In this letter we address the issue of the effects of finite sky coverage for small scale experiments and show that while *in principle* the sample variance can be small, currently

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it is a major source of uncertainty. We find that the naive expectation is essentially correct, namely that the sample variance $\sigma^2_{\rm sam}$ is related to the cosmic variance $\sigma^2_{\rm cos}$ via

$$\sigma^2_{\rm sam} \simeq (4\pi/A)\sigma^2_{\rm cos} \; , \tag{1}$$

where A is the solid angle covered by the experiment.

Consider an (idealized) experiment which is sensitive only to the ℓth multipole. For a Harrison–Zel'dovich spectrum of fluctuations, in general we have $\sigma_{\cos,\ell}/\mu_\ell \sim (\ell + {}^1\!/\!_2)^{-1/2}$, where μ_l is the theoretical r.m.s. value of the ℓth multipole. If this experiment covers N independent patches, each of size $\sim \ell^{-2}$ steradians, then we have

$$\frac{\sigma_{{\rm sam},\ell}}{\mu_\ell} \sim \sqrt{\frac{4\pi/A}{\ell + {}^1\!/\!_2}} \sim \sqrt{\ell/N} \; . \tag{2}$$

Recall that the cosmic variance for a (whole sky) measurement of the quadrupole gives an uncertainty of order the mean. Thus, obtaining a measurement with a theoretical uncertainty below that of the full-sky quadrupole requires $N \gtrsim \ell$.

Since this rule of thumb is not satisfied for most of the current small angular-scale experiments, we were motivated to investigate the effect of the "sample variance" in more detail. We find for the MAX (Meinhold et al. 1993, Gundersen et al 1993) and SP91/ACME (Gaier et al. 1992, Schuster et al. 1993) experiments a theoretical uncertainty of approximately 25–30% in the prediction of $\Delta T/T$ on the appropriate scale. Many authors use a Bayesian technique (e.g. Bond et al. 1991), or a Monte-Carlo simulation (e.g. Górski et al. 1993), in analyzing these experiments. Such techniques automatically account for the sample variance, however sometimes the (dominant) source of the uncertainty is obscured. In the language of Bayesian analysis, our sampling uncertainty indicates that the data from small patches of the sky are highly correlated (the amount of experimental "information" is low), which leads to sensitivity to the "prior distribution" used in the analysis.

We should stress that this sample variance has nothing to do with the experimental precision of the measurements, but simply the fact that they may not necessarily have covered enough of the sky to provide a good estimate of the r.m.s. value of $\Delta T/T$ on the relevant angular scale. In the rest of this letter we develop this argument more rigorously, and give estimates for specific experimental configurations.

Derivation of the sample variance

Consider the correlation function at zero-lag, or $[(\Delta T/T)_{\rm rms}]^2$, as estimated by measurements over a patch of the sky subtending a solid angle A and which we shall call Δ_A. We include the effects of finite beam size and possible "chopping" of the beam in our definition.

The theory predicts a probability distribution for Δ_A, but not its value in our universe. Write C_0 as the mean correlation function at zero-lag: $\langle\Delta_A\rangle = C_0$, where the angled brackets represent an average over an ensemble of "universes". This is a measure of the amplitude of the theoretical power spectrum at the scales probed by the experiment.

Under the assumption that the original fluctuations are gaussian, it is straightforward to calculate

$$\begin{aligned}\left\langle\Delta_A^2\right\rangle &= \frac{1}{A^2}\int_A d\Omega_1 d\Omega_2\,\left\langle\tilde{T}(\hat{n}_1)\tilde{T}(\hat{n}_1)\tilde{T}(\hat{n}_2)\tilde{T}(\hat{n}_2)\right\rangle \\ &= \frac{1}{A^2}\int_A d\Omega_1 d\Omega_2\left[\left\langle\tilde{T}(\hat{n}_1)\tilde{T}(\hat{n}_1)\right\rangle\left\langle\tilde{T}(\hat{n}_2)\tilde{T}(\hat{n}_2)\right\rangle + 2\left\langle\tilde{T}(\hat{n}_1)\tilde{T}(\hat{n}_2)\right\rangle^2\right] .\end{aligned} \tag{3}$$

Here $\tilde{T}(\hat{n})$ refers to the temperature difference assigned to the direction $\hat{n}$ by the experiment, including finite beam width and possible beam "chopping". The angled brackets again represent an average over an ensemble of "universes". To fix the size of the uncertainty in translating from the experimentally measured Δ_A to C_0 we calculate the sample variance, which is simply the second moment about the mean for Δ_A:

$$\begin{aligned}\sigma_{\rm sam}^2 = \Delta_A^{(2)} &= \left\langle\Delta_A^2\right\rangle - \left\langle\Delta_A\right\rangle^2 \\ &= \frac{2}{A^2}\int_A d\Omega_1 d\Omega_2\, C^2(\hat{n}_1\cdot\hat{n}_2)\ .\end{aligned} \tag{4}$$

where $C(\cos\theta)$ is the 2-point correlation function

$$\begin{aligned}C(\hat{n}_1\cdot\hat{n}_2) &\equiv \left\langle\tilde{T}(\hat{n}_1)\tilde{T}(\hat{n}_2)\right\rangle \\ &= \frac{1}{4\pi}\sum_{\ell=2}^{\infty}\left\langle a_\ell^2\right\rangle W_\ell(\hat{n}_1\cdot\hat{n}_2)\ .\end{aligned} \tag{5}$$

Here $a_\ell^2 \equiv \sum_m |a_{\ell m}|^2$, where the $a_{\ell m}$ are the multipole coefficients of the temperature fluctuation on the sky and W_ℓ is the window function appropriate to the experiment under consideration (see Bond et al. 1992, Dodelson & Jubas 1993, White et al. 1993).

One can also derive the third moment about the mean:

$$\Delta_A^{(3)} = \frac{8}{A^3}\int_A d\Omega_1 d\Omega_2 d\Omega_3\, C(\hat{n}_1\cdot\hat{n}_2)\, C(\hat{n}_2\cdot\hat{n}_3)\, C(\hat{n}_3\cdot\hat{n}_1)\ . \tag{6}$$

In general the distribution for Δ_A will be positively skewed. Similar expressions can be written for all the higher moments.

Simple Models

For the purposes of illustration we consider the case of a gaussian autocorrelation function (GACF). Specifically we take

$$C(\cos\theta) = C_0\exp\left[(\cos\theta - 1)/\theta_c^2\right] \simeq C_0\exp\left[-\theta^2/2\theta_c^2\right]\ , \tag{7}$$

and we assume $\theta_c \ll 1$. The correlation function, $C(\cos\theta)$, for both the SP91/ACME and MAX experiments (to choose specific examples) with CDM-like power spectra can be adequately approximated by equation (7). We should emphasize that our $C(\cos\theta)$ has already been convolved with the telescope beam, thus θ_c is a (calculable) function of the power spectrum and the experimental parameters. Often the *unconvolved* $C(\cos\theta)$ is taken to be a GACF, which is *not* a good assumption.

We see from equation (4) that we expect $\Delta_A^{(2)} \sim C_0^2\theta_c^2/A$ for $A \gg \theta_c^2$. To explore this further consider a circular region defined by $0 \le \theta < \theta_0$ and $0 \le \phi < 2\pi$, so that $A = 2\pi(1-\cos\theta_0)$. Using equations (4) and (7) we find that

$$\Delta_A^{(2)} = \frac{2C_0^2}{(1-\cos\theta_0)^2}\int_{\cos\theta_0}^{1} dc_1 dc_2\ \exp\left[2(c_1c_2-1)/\theta_c^2\right] I_0\left(2s_1s_2/\theta_c^2\right) , \tag{8}$$

where c_i and s_i are $\cos\theta_i$ and $\sin\theta_i$ respectively, and I_0 is the modified Bessel function of order zero. We have the two limiting cases

$$\Delta_A^{(2)} \to \begin{cases} 2C_0^2 & \text{as} \quad \theta_0 \to 0\ , \\ \frac{1}{2}\left(\frac{4\pi}{A}\right) C_0^2\theta_c^2 & \text{for} \quad \theta_0 \gg \theta_c\ . \end{cases} \tag{9}$$

We show $\Delta_A^{(2)}$ and $\frac{1}{2}(4\pi/A)C_0^2\theta_c^2$ for $\theta_c = 1^\circ$ in figure 1. The expected result is recovered in the limit of one measurement: for a gaussian random variable x, the variance of x^2 is twice the square of the mean of x^2. Notice that above a few degrees the second limiting form is a good approximation to the exact result. This means that after a few correlation lengths the sample variance, $\Delta_A^{(2)}$, approaches the cosmic variance, $\frac{1}{2}C_0^2\theta_c^2$, as $4\pi/A$, so that there is always something to be gained by sampling more sky.

While the region A defined above clearly illustrates the simple scaling of the sample variance with sky coverage, it is not a good approximation to the scan strategy of SP91/ACME and MAX. To consider these experiments more properly we here make a simplified model of the scan strategies. In both cases we can approximate the scan pattern as a single line of (angular) length α. For our GACF assumption we find

$$\begin{aligned} \Delta_A^{(2)} &= 2\int_0^{\alpha} \frac{d\phi_1}{\alpha}\frac{d\phi_2}{\alpha}\, C^2(\cos|\phi_1-\phi_2|) \\ &\simeq 2C_0^2\,\frac{\theta_c}{\alpha}\left\{\sqrt{\pi}\,\mathrm{erf}\left(\frac{\alpha}{\theta_c}\right) - \frac{\theta_c}{\alpha}\left[1-\exp(-\alpha^2/\theta_c^2)\right]\right\} . \end{aligned} \tag{10}$$

For small α we again recover the limit $\Delta^{(2)} = 2C_0^2$ while for α larger than a few θ_c the second term is small and the error function is approximately unity. Comparing with equation (9) we see that the effective solid angle A for this linear scan is $\sqrt{\pi}\theta_c \times \alpha$. Putting in a MAX correlation angle $\theta_c \simeq 1/2^\circ$ and a scan length of $\alpha = 6^\circ$ we find that $\sigma_{\rm sam} = \sqrt{\Delta_A^{(2)}} \simeq 0.5C_0$; that is, the uncertainty is approximately 50% of the mean $[(\Delta T/T)_{\rm rms}]^2 = \langle\Delta_A\rangle = C_0$. In the case of SP91/ACME, $\theta_c \simeq 2^\circ$ and $\alpha \simeq 9\times 2\overset{\circ}{.}1$, giving a relative error of approximately 60%. The actual value of θ_c depends on the precise cosmological model assumed, but the values used above are representative for CDM and these experiments.

The implication of this simple analysis is that the theoretical predictions for $\Delta T/T$ for the experiments considered should be assigned a relative error of approximately 25–30% due to the "sample variance". (However recall that the distribution of Δ_A will be non-gaussian; see e.g. Scaramella & Vittorio 1991, Cayon et al. 1991, White et al. 1993.) Other experiments have similarly non-negligible sample variances because of incomplete sky coverage. This "range" of theoretical predictions should be considered in comparing theory with small scale experiments. Only COBE and MIT approach the unavoidable cosmic variance limit for the scales which they probe. Note, however, that the galactic cut made by the COBE team ($|b| > 20°$) increases the relative error on $\Delta T/T$ by 23% over what it would be for a whole sky measurement.

Conclusions

Our analysis makes clear that the potential gain in theoretical precision on smaller angular scales is in fact only realized once a significant fraction of the sky has been covered. Thus the effects of finite sky coverage are serious for all of the current small scale experiments: VLA, ATCA, OVRO, MSAM, MAX, SP91/ACME and (on larger scales) Tenerife. Including the "sample variance" in the theoretical predictions for these experiments may help to reconcile the apparently contradictory results being reported on small angular scales (Gaier et al. 1992, Schuster et al. 1993, Gundersen et al. 1993, Meinhold et al. 1993). Current analyses of the data (including Monte-Carlo and Bayesian analyses) correctly take into account the "sample variance", but provide little physical insight or guide to the size of the effects. We hope that our approach is sufficiently straightforward that it will make the comparison of theory and measurement more transparent for these small scales, and perhaps also act as an aid for the design of future experiments.

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Figure Caption

Figure: The sample variance, $\Delta_A^{(2)}$, from equation (8) as a function of sky coverage, θ_0, for a GACF with $\theta_c = 1°$ (solid line). Also shown is the naive expectation $(4\pi/A)\sigma_{\rm cos}^2$ (dashed line) where $A = 2\pi(1 - \cos\theta_0)$ is the solid angle covered by the region sampled and $\sigma_{\rm cos}^2 = \frac{1}{2}C_0^2\theta_c^2$ is the cosmic variance. This is a good approximation to the full result above a few correlation lengths.

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